Probability Problem Set

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1. **De Morgan's Laws** Let A and B both be events in Ω, the sample space.

   (a) Prove that $(A \cap B)^C = A^C \cup B^C$.

   (b) Prove that $(A \cup B)^C = A^C \cap B^C$.

2. **Hansen 1.3 Adapted** From a 52-card deck of playing cards, draw five cards to make a hand.

   (a) Let A be the event that the hand has exactly two Kings. Find $P(A)$.

   (b) Let B be the event the hand is a straight (not including straight flushes). Find $P(B)$.

   (c) Let C be the event that the hand is a flush (not including straight flushes). Find $P(C)$.

3. **Hansen 1.6** Can A and B be disjoint if $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$? Prove your answer.

4. **Hansen 1.17** Suppose 1% of athletes use banned steroids and that a drug detection test has an accuracy rate of 40%. It also has a false positive rate of 1%. If an athlete tests positive, what is the conditional probability that the athlete has taken banned steroids?

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5. **The Monte Hall Problem** You are on the game show "Let's Make a Deal with Monte Hall." There are three doors in front of you: doors A, B, and C. Your goal is to select the door with the prize behind it. Assume, without loss of generality, that you select door A. Monte then opens one of the other two doors, say door B, revealing that there is no prize behind it. He then gives you the option to switch your choice of doors. Should you stick with door A or switch your choice to door C? Assume that the *ex ante* probability of the prize being behind each door is 1/3.

6. **The St. Petersburg Paradox** Suppose a wealthy billionaire runs a game. Each participant gets to flip a fair coin until the coin comes up as heads for the first time. At that point, the participant wins $\$2^n$, where n denotes the number of flips. The billionaire charges \$1 billion to play.

   (a) Write down the pmf and verify that it is valid.

   (b) What is the probability that you win more than \$4?

   (c) Suppose you are an economic agent that makes decisions based only on your expected payoff. Should you pay the entry fee?

7. **Hansen 2.6** Compute $\mathbb{E}[X]$ and $Var(X)$ for the following distributions:

   (a) $f(x) = ax^{-a-1}$, for $0 < x < 1$ and $a > 0$.

   (b) $f(x) = \frac{1}{n}$, for $x = 1, 2, ..., n$. Hint: show by induction that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and that $\sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}$.

   (c) $f(x) = \frac{3}{2}(x-1)^2$, for $0 < x < 2$.

8. **Hansen 2.7** Let X have density:

$$f_X(x) = \frac{1}{2^{r/2}\Gamma(r/2)} x^{r/2-1} e^{-x/2}$$

for $x > 0$. Let $Y = 1/X$. Derive the density of Y for $y > 0$.

9. **Hansen 3.4 Adapted** Suppose $X \sim Unif[a, b]$.

   (a) Show that the pdf of X is a valid pdf.

   (b) Find the expected value of X.

   (c) Find the variance of X.

10. **Joint Distributions** Suppose that the joint pdf of X and Y is:

$$f(x, y) = \frac{1}{4}$$

and that $-1 < X < 1$ and $-1 < Y < 1$.

   (a) Show that this is a valid pdf (Hint: integrate over both X and Y. What do all valid pdfs integrate to?).

   (b) We can find the pdf of X by integrating out Y from the joint distribution. Find $f(x)$.

   (c) We can find the pdf of Y similarly. Find $f(y)$.

   (d) We said that two random variables are independent if $P(A \cap B) = P(A)P(B)$. Are X and Y independent here?

   (e) Find the conditional distribution $f(y|x)$ (Hint: the formula is the same as the probability formula).

   (f) Find $\mathbb{E}[Y|X]$ (Hint: $g(y) = y$ here. Use the conditional pdf.).

   (g) Find $\mathbb{E}[Y]$. Compare your answer here to your answer in (f). Are you surprised? Explain why or why not.